

On Pattern Avoiding Alternating Permutations

Joanna N. Chen¹, William Y.C. Chen², Robin D.P. Zhou³

^{1,3}Center for Combinatorics, LPMC-TJKLC
Nankai University
Tianjin 300071, P.R. China

²Center for Applied Mathematics
Tianjin University
Tianjin 300072, P.R. China

¹joanna@cfc.nankai.edu.cn, ²chenyc@tju.edu.cn, ³robin@cfc.nankai.edu.cn

Abstract

An alternating permutation of length n is a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ such that $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots$. Let A_n denote set of alternating permutations of $\{1, 2, \dots, n\}$, and let $A_n(\sigma)$ be set of alternating permutations in A_n that avoid a pattern σ . Recently, Lewis used generating trees to enumerate $A_{2n}(1234)$, $A_{2n}(2143)$ and $A_{2n+1}(2143)$, and he posed several conjectures on the Wilf-equivalence of alternating permutations avoiding certain patterns. Some of these conjectures have been proved by Bóna, Xu and Yan. In this paper, we prove the two relations $|A_{2n+1}(1243)| = |A_{2n+1}(2143)|$ and $|A_{2n}(4312)| = |A_{2n}(1234)|$ as conjectured by Lewis.

Keywords: alternating permutation, pattern avoidance, generating tree

AMS Subject Classifications: 05A05, 05A15

1 Introduction

The objective of this paper is to prove two conjectures of Lewis on the Wilf-equivalence of alternating permutations avoiding certain patterns of length four.

We begin with some notation and terminology. Let $[n] = \{1, 2, \dots, n\}$, and let S_n be the set of permutations of $[n]$. A permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is said to be an alternating permutation if $\pi_1 < \pi_2 > \pi_3 < \pi_4 > \cdots$. An alternating permutation is also called an up-down permutation. A permutation π is said to be a down-up permutation if $\pi_1 > \pi_2 < \pi_3 > \pi_4 < \cdots$. We denote by A_n and A'_n the set of alternating permutations and the set of down-up permutations of $[n]$, respectively. For a permutation $\pi \in S_n$, its reverse $\pi^r \in S_n$ is defined by $\pi^r(i) = \pi(n+1-i)$ for $1 \leq i \leq n$. The complement of π , denoted $\pi^c \in S_n$, is defined by $\pi^c(i) = n+1-\pi(i)$ for $1 \leq i \leq n$. It is clear that the complement operation gives a bijection between A_n and A'_n .

Given a permutation π in S_n and a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_k \in S_k$, where $k \leq n$, we say that π contains a pattern σ if there exists a subsequence $\pi_{i_1}\pi_{i_2}\cdots\pi_{i_k}$ ($1 \leq i_1 < i_2 < \cdots < i_k \leq n$) of π that is order isomorphic to σ , in other words, for all $l, m \in [k]$, we have $\pi_{i_l} < \pi_{i_m}$ if and only if $\sigma_l < \sigma_m$. Otherwise, we say that π avoids a pattern σ , or π is σ -avoiding. For example, 74538126 is 1234-avoiding, while it contains pattern 3142 corresponding to the subsequence 7486.

Let $S_n(\sigma)$ denote the set of permutations of length n that avoid a pattern σ . Let $A_n(\sigma)$ denote the set of σ -avoiding alternating permutations of $[n]$, and let $A'_n(\sigma)$ denote the set of σ -avoiding down-up permutations of $[n]$. Mansour [7] showed that $|A_{2n}(132)| = C_n$, where C_n is the Catalan number

$$\frac{1}{n+1} \binom{2n}{n}.$$

Meanwhile, Deutsch and Reifegerste (as reported by Stanley [9]) showed that $|A_{2n}(123)| = C_n$. Recently, Lewis [6] showed that the generating trees for $A_{2n}(1234)$ and $A_{2n}(2143)$ are isomorphic to the generating tree for the set of standard Young tableaux of shape (n, n, n) . From the hook-length formula it follows that

$$|A_{2n}(1234)| = |A_{2n}(2143)| = \frac{2(3n)!}{n!(n+1)!(n+2)!}. \quad (1.1)$$

The above number is called the n -th 3-dimensional Catalan number, and we shall denote it by $C_n^{(3)}$. Notice that $C_n^{(3)}$ also equals the number of walks in 3-dimensions from $(0, 0, 0)$ to (n, n, n) by using steps $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ that do not go below the plane $x = y = z$. Lewis showed that $A_{2n+1}(2143)$ has the same generating tree as that of shifted standard Young tableaux of shape $(n+2, n+1, n)$. Using the hook-length formula for shifted standard Young tableaux given by Krattenthaler [1], we deduce that

$$|A_{2n+1}(2143)| = \frac{2(3n+3)!}{n!(n+1)!(n+2)!(2n+1)(2n+2)(2n+3)}.$$

The following conjectures were posed by Lewis [6].

Conjecture 1.1 For $n \geq 1$ and $\sigma \in \{1243, 2134, 1432, 3214, 2341, 4123, 3421, 4312\}$, we have

$$|A_{2n}(\sigma)| = |A_{2n}(1234)| = |A_{2n}(2143)|.$$

Conjecture 1.2 For $n \geq 0$ and $\sigma \in \{2134, 4312, 3214, 4123\}$, we have

$$|A_{2n+1}(\sigma)| = |A_{2n+1}(1234)|.$$

Conjecture 1.3 For $n \geq 0$ and $\sigma \in \{1243, 3421, 1432, 2341\}$, we have

$$|A_{2n+1}(\sigma)| = |A_{2n+1}(2143)|.$$

By showing that a classical bijection on pattern avoiding permutations preserves the alternating property, Bóna [2] proved that

$$|A_{2n}(1243)| = |A_{2n}(1234)|, \quad (1.2)$$

$$|A_{2n+1}(2134)| = |A_{2n+1}(1234)|. \quad (1.3)$$

Xu and Yan [11] constructed bijections that lead to the following relations

$$|A_{2n}(4123)| = |A_{2n}(1432)| = |A_{2n}(1234)|,$$

$$|A_{2n+1}(1432)| = |A_{2n+1}(2143)|,$$

$$|A_{2n+1}(4123)| = |A_{2n+1}(1234)|.$$

As for the above conjectures, there are essentially two unsolved cases, namely,

$$|A_{2n+1}(1243)| = |A_{2n+1}(2143)|, \quad (1.4)$$

and

$$|A_{2n}(4312)| = |A_{2n}(1234)|, \quad (1.5)$$

because the other remaining cases can be deduced by the reverse and complement operations.

In this paper, we prove the above conjectures (1.4) and (1.5). To be more specific, we show that the generating tree for $A_{2n+1}(1243)$ coincides with the generating tree for $A_{2n+1}(2143)$ as given by Lewis [6]. So we are led to relation (1.4). The construction of the generating tree for $A_{2n+1}(1243)$ can be adapted to obtain the generating tree for $A_{2n}(1243)$, which turns out to be isomorphic to the generating tree for $A_{2n}(1234)$ as constructed by Lewis [6]. This gives another proof of relation (1.2) conjectured by Lewis and proved by Bóna.

To prove (1.5), we show that the generating tree for $A_{2n+1}(1243)$ is isomorphic to the generating tree for the set of shifted standard Young tableaux of shape $(n+2, n+1, n)$ as given by Lewis [6]. We adopt the notation $SHSYT(\lambda)$ for the set of shifted standard Young tableaux of shape λ . As can be easily seen, a label (a, b) in the generating tree for $A_{2n+1}(1243)$ corresponds to a label $(a+1, b)$ in the generating tree for $SHSYT(n+2, n+1, n)$. By restricting the correspondence to certain labels of the generating trees, we obtain a bijection between a subset of $A_{2n+1}(1243)$ and a subset of $SHSYT(n+2, n+1, n)$. This leads to the relation $|A_{2n}(4312)| = |SHSYT(n+2, n, n-2)|$. By the hook-length formula for shifted standard Young tableaux, we see that $SHSYT(n+2, n, n-2)$ is counted by $C_n^{(3)}$. Since $A_{2n}(1234)$ is also enumerated by $C_n^{(3)}$, we arrive at relation (1.5).

Since we already have that $|A_{2n}(4312)| = |A_{2n}(1234)|$, it is natural to consider whether one can construct a generating tree for $A_{2n}(4312)$ that is isomorphic to the generating tree for $A_{2n}(1234)$ given by Lewis. While we have not found such a generating tree for $A_{2n}(4312)$, we obtain a generating tree for $A_{2n}(4312)$ that can be used to give a second proof of relation (1.4). By deleting the leaves of the generating tree for

$A_{2n}(4312)$ and changing the label (a, b) to $(a - 1, b)$, we are led to the generating tree for $A_{2n}(3412)$ as given by Lewis [6]. Furthermore, by restricting this correspondence to certain labels, we obtain relation (1.4).

This paper is organized as follows. In Section 2, we construct a generating tree for $A_{2n+1}(1243)$, which turns out to be the same with the generating tree for $A_{2n+1}(2143)$ given by Lewis. This proves relation (1.4). By similar constructions, we see that $A_{2n}(1234)$ and $A_{2n}(1243)$ have isomorphic generating trees. This yields another proof of (1.2). In Section 3, we prove (1.5) by showing that $|A_{2n}(4312)|$ is equal to the number of shifted standard Young tableaux of shape $(n + 2, n, n - 2)$. In Section 4, we construct a generating tree for $A_{2n}(4312)$, and give another proof of (1.4).

2 Generating trees for $A_{2n+1}(1243)$ and $A_{2n}(1243)$

In this section, we construct the generating tree for $A_{2n+1}(1243)$ which turns out to be the same as that for $A_{2n+1}(2143)$. This proves (1.4), which we restate as the following theorem.

Theorem 2.1 *For $n \geq 0$, we have $|A_{2n+1}(1243)| = |A_{2n+1}(2143)|$.*

We also obtain a generating tree for $A_{2n}(1243)$ and we show that it is isomorphic to the generating tree for $A_{2n}(1234)$. This confirms (1.2), which we restate as the following theorem.

Theorem 2.2 *For $n \geq 1$, we have $|A_{2n}(1243)| = |A_{2n}(1234)|$.*

Let us give an overview of the terminology on generating trees. Given a sequence $\{\Sigma_n\}_{n \geq 1}$ of finite, nonempty sets with $|\Sigma_1| = 1$, a generating tree for this sequence is a rooted, labeled tree such that the vertices at level n are the elements of Σ_n and the label of each vertex determines the multiset of labels of its children. Thus, the generating tree is fully described by its root vertex and the succession rule $L \rightarrow S$ which gives the set S of labels of the children in terms of the label L of their parent. Here, we denote a generating tree in the following form,

$$\begin{cases} \text{root:} & \text{the label of the root,} \\ \text{rule:} & \text{succession rules.} \end{cases}$$

Sometimes we also refer a generating tree for Σ_n to the generating tree for the sequence $\{\Sigma_n\}_{n \geq 1}$.

The construction of a generating tree for $\{\Sigma_n\}_{n \geq 1}$ requires the generation of Σ_{n+1} based on Σ_n . For $u \in \Sigma_n$, $w \in \Sigma_{n+1}$, let w be a child of u in the generating tree if

and only if w is generated by u . Thus it is sufficient to determine the structure of the generating tree by defining the children of each element.

To illustrate the idea of generating trees, we consider the construction of a generating tree for S_n . We need to determine the children of each permutation in S_n . Given $\pi \in S_n$, we can generate $n + 1$ permutations in S_{n+1} . For $1 \leq i \leq n + 1$, let $i \mapsto \pi$ denote the permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{n+1}$ in S_{n+1} such that $\sigma_1 = i$ and $\sigma_2 \sigma_3 \cdots \sigma_{n+1}$ is order isomorphic to π . In other words, $i \mapsto \pi$ is the permutation obtained from π by adding i to the beginning of π and increasing each element not less than i by 1. For example, $3 \mapsto 3142 = 34152$ is a child of 3142 in the generating tree.

Notice that Lewis [6] used the notation $\pi \leftarrow i$ denote the permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_{n+1}$ in S_{n+1} such that $\sigma_{n+1} = i$ and $\sigma_1 \sigma_2 \cdots \sigma_n$ order isomorphic to π . The idea of generating trees is to give succession rules for the structure of the generating tree by assigning labels to the vertices. For the case of permutations, given $\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$, we associate it with a label (π_1, n) . Then we have the generating tree for S_n as follows

$$\begin{cases} \text{root:} & (1, 1), \\ \text{rule:} & (i, n) \rightarrow \{(j, n + 1) \mid 1 \leq j \leq n + 1\}. \end{cases}$$

By the recursive construction of alternating permutations, Lewis [6] obtained generating schemes for A_{2n} and $A_{2n}(\sigma)$. Here we describe the recursive constructions of A_{2n} and $A_{2n}(\sigma)$ by adding elements at the beginning. This choice of notation seems to be more convenient for the description of the construction of the generating trees for $A_{2n+1}(1243)$ and $A_{2n}(4312)$.

For $n \geq 1$, let $u = u_1 u_2 \cdots u_{2n}$ be an alternating permutation in A_{2n} . The generating tree is constructed based on the following generating scheme. Consider alternating permutations $w = w_1 w_2 w_3 \cdots w_{2n+2}$ in A_{2n+2} such that $w_3 w_4 \cdots w_{2n+2}$ is order isomorphic to u . Such permutations are set to be the children of u in the generating tree. One can also use this recursive procedure to construct pattern avoiding alternating permutations. To be specific, given $u \in A_{2n}(\sigma)$, the set of the children of u is precisely the set $\{w \mid w = v_1 \mapsto (v_2 \mapsto u), w \in A_{2n+2}(\sigma)\}$. The generating scheme for pattern avoiding alternating permutations of odd length can be constructed in the same manner.

We now proceed to construct the generating trees for $A_{2n+1}(1243)$ and $A_{2n}(1243)$. In fact, these two sets have the same succession rules with different roots. Here we shall only present the derivation of the succession rules for $A_{2n+1}(1243)$. To this end, we need to characterize the set of 1243-avoiding alternating permutations in A_{2n+3} that are generated by an alternating permutation u in $A_{2n+1}(1243)$. Such a characterization leads to a labeling along with succession rules.

Theorem 2.3 *For $n \geq 0$, given a permutation $u = u_1 u_2 \cdots u_{2n+1} \in A_{2n+1}(1243)$, define*

$$f(u) = \max\{0, u_j \mid \text{there exists } i \text{ such that } i < j \text{ and } u_i > u_j\},$$

$$e(u) = \max\{0, u_i \mid \text{there exist } j \text{ and } k \text{ such that } i < j < k \text{ and } u_i < u_k < u_j\}.$$

Then w is a child of u if and only if it is of the form $w = v_1 \mapsto (v_2 \mapsto u)$, where

$$e(u) < v_1 \leq v_2, \quad (2.1)$$

and

$$\max\{u_1 + 1, f(u) + 1\} \leq v_2 \leq 2n + 2. \quad (2.2)$$

Proof. Suppose $w = w_1 w_2 \cdots w_{2n+3}$ is a child of u , by definition, w is of the form $v_1 \mapsto (v_2 \mapsto u)$ and $w \in A_{2n+3}(1243)$. Since w is alternating on $[2n + 3]$, we have $v_1 \leq v_2 \leq 2n + 2$ and $v_2 \geq u_1 + 1$. By the order of the insertions of v_1 and v_2 , we see that $w_1 = v_1$ and $w_2 = v_2 + 1$. Since w is 1243-avoiding, we have $v_2 \geq f(u) + 1$; Otherwise, there exists $i < j$ such that $u_i > u_j$ and $v_2 \leq u_j$. This implies that $w_1 w_2 w_{i+2} w_{j+2} = v_1 (v_2 + 1) (u_i + 2) (u_j + 2)$ forms a 1243-pattern. Moreover, we have $v_1 > e(u)$; Otherwise, there exist $i < j < k$ such that $u_i < u_k < u_j$ and $v_1 \leq u_i$. Clearly, $w_{i+2} > u_i \geq v_1$. Thus $w_1 w_{i+2} w_{j+2} w_{k+2}$ is of pattern 1243. So we are led to the relations (2.1) and (2.2).

Conversely, we assume that v_1 and v_2 are integers satisfying conditions (2.1) and (2.2). We wish to show that $w = v_1 \mapsto (v_2 \mapsto u)$ is an alternating permutation in $A_{2n+3}(1243)$, that is, w is a child of u in the generating tree. It is evident from (2.1) and (2.2) that $v_1 \leq v_2$ and $v_2 > u_1$. So we have $w_1 < w_2 > w_3$, from which we see that w is alternating since $w_3 w_4 \cdots w_{2n+3}$ is order isomorphic to u .

It remains to show that w is 1243-avoiding. Assume to the contrary that w contains a 1243-pattern, that is, there exist $t < i < j < k$ such that $w_t w_i w_j w_k$ is of pattern 1243. We claim that $t = 1$ or 2 . Otherwise, we assume that $t \geq 3$. Since $u_{t-2} u_{i-2} u_{j-2} u_{k-2}$ is isomorphic to $w_t w_i w_j w_k$, we find $u_{t-2} u_{i-2} u_{j-2} u_{k-2}$ forms a 1243-pattern, which is a contradiction. It follows that $t \leq 2$. If $w_2 w_i w_j w_k$ forms a 1243-pattern, then $w_1 w_i w_j w_k$ is also a 1243-pattern. Hence we can always choose $t = 1$. To prove w is 1243-avoiding, it is sufficient to show that it is impossible for $w_1 w_i w_j w_k$ to be a 1243-pattern.

We now assume that $w_1 w_i w_j w_k$ is a 1243-pattern. If $i = 2$, we have $w_2 < w_k$. Since $w_2 = v_2 + 1$ and $w_k \leq u_{k-2} + 2$, we get $v_2 \leq u_{k-2}$. Note that $u_{j-2} u_{k-2}$ is order isomorphic to $w_j w_k$, so we have $u_{j-2} > u_{k-2}$. By the definition of $f(u)$, we find $u_{k-2} \leq f(u)$. It follows that $v_2 \leq f(u)$, which contradicts to the fact that $v_2 \geq f(u) + 1$. Hence we have $i > 2$.

We now claim that $w_1 \leq u_{i-2}$. Otherwise, we assume $w_1 > u_{i-2}$. Since $w_1 = v_1$ and $v_1 \leq v_2$, we find that $u_{i-2} < v_1 \leq v_2$. By the construction of w , we have $w_i = u_{i-2}$. This yields $w_i < w_1$, which contradicts to the assumption that $w_1 w_i w_j w_k$ is a 1243-pattern. This proves the claim. Clearly, $u_{i-2} u_{j-2} u_{k-2}$ is a 132-pattern since it is order isomorphic to $w_i w_j w_k$. By the definition of $e(u)$, we get $u_{i-2} \leq e(u)$. Thus $v_1 = w_1 \leq u_{i-2} \leq e(u)$, which contradicts to the fact $v_1 > e(u)$. So we reach the conclusion that the assumption that $w_1 w_i w_j w_k$ is a 1243-pattern is not valid. In other words, w is 1243-avoiding. This completes the proof. \blacksquare

In light of the above characterization, we are led to a labeling scheme for alternating permutations in $A_{2n+1}(1243)$. For $u \in A_{2n+1}(1243)$, we associate a label (a, b) to u , where

$$a = 2n + 2 - \max\{u_1 + 1, f(u) + 1\}, \quad (2.3)$$

$$b = 2n + 2 - e(u). \quad (2.4)$$

For example, the permutation $1 \in A_1(1243)$ has label $(0, 2)$, and the permutation $2546173 \in A_7(1243)$ has label $(3, 6)$.

The above labeling scheme enables us to derive succession rules for $A_{2n+1}(1243)$. Recall that the functions $f(u)$ and $e(u)$ for a permutation $u \in A_{2n+1}(1243)$ are defined in Theorem 2.3. In fact, these functions can be defined on a permutation on any finite set of positive integers. For example, let $t = 48152967$, which is a permutation on $\{1, 2, 4, 5, 6, 7, 8, 9\}$, we have $f(t) = 7$ and $e(t) = 5$.

Theorem 2.4 *For $n \geq 0$, given $u = u_1 u_2 \cdots u_{2n+1} \in A_{2n+1}(1243)$ with label (a, b) , the set of labels of the children of u is given by the set*

$$\{(x, y) \mid 1 \leq x \leq a + 1, x + 2 \leq y \leq b + 2\}.$$

Proof. Assume that $w = v_1 \mapsto (v_2 \mapsto u)$ is a child of u and we write $w = w_1 w_2 \cdots w_{2n+3}$. We aim to determine the range of the label (x, y) of w . According to Theorem 2.3, we have relations (2.1) and (2.2), namely, $e(u) < v_1 \leq v_2$ and $\max\{u_1 + 1, f(u) + 1\} \leq v_2 \leq 2n + 2$. Since $w \in A_{2n+3}(1243)$, from the labeling schemes (2.3) and (2.4) it follows that

$$x = 2n + 4 - \max\{w_1 + 1, f(w) + 1\},$$

$$y = 2n + 4 - e(w).$$

We proceed to compute $f(w)$ and $e(w)$. Notice that the insertions of v_1 and v_2 to u may cause new 21-patterns and 132-patterns. Let $s = w_3 w_4 \cdots w_{2n+3}$. To determine $f(w)$, it suffices to compare $f(s)$ with the smaller element in each new 21-pattern. Similarly, $e(w)$ can be obtained by comparing $e(s)$ with the smallest element in each new 132-pattern. Here are two cases.

Case 1: $e(u) < v_1 < v_2$. It is clear that s is order isomorphic to u . We claim that $e(u) = e(s)$. We first consider the case $e(u) = 0$. In this case, by definition we see that u is 132-avoiding. Thus, s is also 132-avoiding, namely, $e(s) = 0$. We now turn to the case $e(u) \neq 0$. In other words, $e(u) = u_i$ for some $1 \leq i \leq 2n + 1$. Since s is order isomorphic to u , we deduce that $e(s) = s_i$. Hence $u_i = e(u) < v_1 < v_2$. Since $w = v_1 \mapsto (v_2 \mapsto u)$, we see that $s_i = u_i$. This yields $e(s) = e(u) = u_i$. So the claim is verified.

To compute $e(w)$, we consider the new 132-patterns caused by the insertions of v_1 and v_2 into u . Since $w_2 = v_2 + 1$ and $v_2 > f(u)$, by the definition of $f(u)$, we find that w_2 cannot appear as the smallest entry of any 132-pattern of w . Hence we need only to

consider the new 132-patterns caused by the insertion of v_1 into u . Since $v_1(v_2 + 1)v_2$ is a 132-pattern and $v_1 > e(u) = e(s)$, we conclude that $e(w) = \max\{v_1, e(s)\} = v_1$.

To compute $f(w)$, we first determine $f(s)$. There are two cases. If $e(u) < v_1 \leq f(u)$, then $f(u) \neq 0$. Hence $f(u) = u_i$ for some $1 \leq i \leq 2n + 1$. Since s is order isomorphic to u , we have $f(s) = s_i$. Noting that $v_2 > f(u) \geq v_1$, we get $s_i = u_i + 1$. It follows that $f(s) = f(u) + 1$. If $f(u) < v_1 < v_2$, by the above argument for determining $e(w)$, we have $f(s) = f(u)$. Therefore, both cases we have $f(s) \leq f(u) + 1$.

We now consider the new 21-patterns caused by the insertions of v_1 and v_2 into u . Since $v_1 < v_2$ and $w_2 = v_2 + 1$, we see that $(v_2 + 1)v_2$ is a 21-pattern of w . Moreover, it can be seen that v_2 is the largest entry among the smaller elements of the newly formed 21-patterns. From the fact that $f(s) \leq f(u) + 1 \leq v_2$ we deduce that $f(w) = \max\{v_2, f(s)\} = v_2$. Therefore, we have

$$\begin{aligned} x &= 2n + 4 - \max\{w_1 + 1, f(w) + 1\} = 2n + 3 - v_2, \\ y &= 2n + 4 - e(w) = 2n + 4 - v_1. \end{aligned}$$

Note that $e(u) < v_1 < v_2$ and $\max\{u_1 + 1, f(u) + 1\} \leq v_2 \leq 2n + 2$, we obtain

$$\begin{aligned} 1 &\leq x \leq 2n + 3 - \max\{u_1 + 1, f(u) + 1\}, \\ 2n + 5 - v_2 &\leq y \leq 2n + 3 - e(u). \end{aligned}$$

By the labeling rule (2.3), namely, $a = 2n + 2 - \max\{u_1 + 1, f(u) + 1\}$, we deduce that $1 \leq x \leq a + 1$. By the labeling rule (2.4), namely, $b = 2n + 2 - e(u)$ and the fact that $x = 2n + 3 - v_2$, we see that $x + 2 \leq y \leq b + 1$. It is easily checked that (x, y) can be any pairs of integers such that $1 \leq x \leq a + 1$ and $x + 2 \leq y \leq b + 1$, since $e(u) < v_1 < v_2$ and $\max\{u_1 + 1, f(u) + 1\} \leq v_2 \leq 2n + 2$. This implies that the set of labels of the children of u considered in this case is given by

$$\{(x, y) \mid 1 \leq x \leq a + 1 \text{ and } x + 2 \leq y \leq b + 1\}.$$

Case 2: $v_1 = v_2$. By (2.2), we have $v_1 = v_2 > f(u)$. Since s is order isomorphic to u , by the argument for computing $e(w)$ in Case 1, we find $f(s) = f(u)$. Let us analyze the new 21-patterns caused by the insertions of v_1 and v_2 into u . Clearly, $(v_2 + 1)(v_2 - 1)$ is a 21-pattern of w . Moreover, it is obvious that $v_2 - 1$ is the largest entry of the smaller elements in the newly formed 21-patterns. Since $v_2 - 1 \geq f(u) = f(s)$, we have $f(w) = \max\{v_2 - 1, f(s)\} = v_2 - 1$.

By (2.1), namely, $e(u) < v_1 \leq v_2$, and the fact that s is order isomorphic to u , it can be seen that $e(s) = e(u)$. Since $v_1 = v_2 > f(u)$, the insertions of v_1 and v_2 do not create any new 132-patterns. It yields that $e(w) = e(s) = e(u)$. Consequently, we get

$$\begin{aligned} x &= 2n + 4 - \max\{w_1 + 1, f(w) + 1\} = 2n + 3 - v_2, \\ y &= 2n + 4 - e(w) = 2n + 4 - e(u). \end{aligned}$$

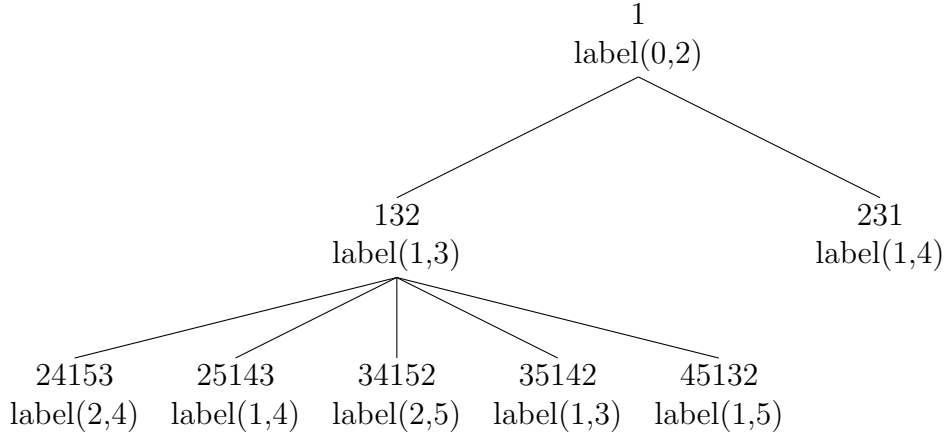


Figure 2.1: The first few levels of the generating tree for $A_{2n+1}(1243)$

We claim that $\max\{u_1 + 1, f(u) + 1\} \leq v_1 = v_2 \leq 2n + 2$. From the definitions of $f(u)$ and $e(u)$, it can be seen that $f(u) \geq e(u)$, since each 132-pattern contains a 21-pattern. Thus the claim follows from (2.1) and (2.2). This implies that

$$1 \leq x \leq 2n + 3 - \max\{u_1 + 1, f(u) + 1\}.$$

By the labeling rule (2.3), namely, $a = 2n + 2 - \max\{u_1 + 1, f(u) + 1\}$, we deduce that $1 \leq x \leq a + 1$. By the labeling rule (2.4), namely, $b = 2n + 2 - e(u)$, we get $y = b + 2$.

Using the same argument as in Case 1, we see that (x, y) range over all pairs of integers such that $1 \leq x \leq a + 1$ and $y = b + 2$. Hence the set of labels of the children of u considered in this case is given by

$$\{(x, y) \mid 1 \leq x \leq a + 1 \text{ and } y = b + 2\}.$$

Combining Case 1 and Case 2, the set of labels of the children of u is given by

$$\{(x, y) \mid 1 \leq x \leq a + 1, x + 2 \leq y \leq b + 2\},$$

as required. This completes the proof. ■

Indeed, the above characterization of the labels of the children of a permutation u in $A_{2n+1}(1243)$ implies that the label of a child w of u is uniquely determined by w . Thus, in the representation of the generating tree for $A_{2n+1}(1243)$ we may only keep the labels and ignore the alternating permutations themselves. The generating tree can be described as follows:

$$\begin{cases} \text{root:} & (0, 2), \\ \text{rule:} & (a, b) \mapsto \{(x, y) \mid 1 \leq x \leq a + 1 \text{ and } x + 2 \leq y \leq b + 2\}. \end{cases} \quad (2.5)$$

Figure 2.1 gives the first few levels of the generating tree for $A_{2n+1}(1243)$.

Comparing the above description of the generating tree for $A_{2n+1}(1243)$ and the generating tree for $A_{2n+1}(2143)$ as given by Lewis [6], we arrive at the assertion that there is a bijection between $A_{2n+1}(1243)$ and $A_{2n+1}(2143)$. This proves Theorem 2.1.

The construction of the generating tree for $A_{2n+1}(1243)$ can be easily adapted to derive the generating tree for $A_{2n}(1243)$. The following theorem gives a similar characterization of the set of 1243-avoiding alternating permutations in A_{2n+2} that are generated by an alternating permutation u in $A_{2n}(1243)$.

Theorem 2.5 *For $n \geq 1$, given a permutation $u = u_1 u_2 \cdots u_{2n} \in A_{2n}(1243)$, then w is a child of u if and only if it is of the form $w = v_1 \mapsto (v_2 \mapsto u)$, where*

$$\begin{aligned} e(u) &< v_1 \leq v_2, \\ \max\{u_1 + 1, f(u) + 1\} &\leq v_2 \leq 2n + 1. \end{aligned}$$

Based on the above characterization, we assign a label (a, b) to an alternating permutation $u = u_1 u_2 \cdots u_{2n} \in A_{2n}(1243)$, where

$$\begin{aligned} a &= 2n + 1 - \max\{u_1 + 1, f(u) + 1\}, \\ b &= 2n + 1 - e(u). \end{aligned}$$

The succession rules for $A_{2n}(1243)$ are exactly the same as these for $A_{2n+1}(1243)$, since they do not depend on the parity of the length of u . Note that $12 \in A_{2n}(1243)$ has label $(1, 3)$. So the generating tree for $A_{2n}(1243)$ can be described as follows:

$$\begin{cases} \text{root:} & (1, 3), \\ \text{rule:} & (a, b) \mapsto \{(x, y) \mid 1 \leq x \leq a + 1 \text{ and } x + 2 \leq y \leq b + 2\}. \end{cases} \quad (2.6)$$

Recall the following generating tree for $A_{2n}(1234)$ given by Lewis [6]:

$$\begin{cases} \text{root:} & (2, 3), \\ \text{rule:} & (a, b) \mapsto \{(x, y) \mid 2 \leq x \leq a + 1 \text{ and } x + 1 \leq y \leq b + 2\}. \end{cases} \quad (2.7)$$

Apparently, the above two generating trees are isomorphic via the correspondence $(a, b) \rightarrow (a + 1, b)$. This gives another proof of Theorem 2.2, which was proved by Bóna [2] via a direct bijection.

3 Proof of the conjecture $|A_{2n}(4312)| = |A_{2n}(1234)|$

In this section, we prove the following theorem which leads to the relation $|A_{2n}(4312)| = |A_{2n}(1234)|$ conjectured by Lewis [6].

Theorem 3.1 *For $n \geq 3$, we have $|A_{2n}(4312)| = |SHSYT(n+2, n, n-2)|$.*

Recall that $|SHSYT(n+2, n, n-2)|$ is known as the n -th 3-dimensional Catalan number $C_n^{(3)}$. As Lewis [6] has proved that $|A_{2n}(1234)| = C_n^{(3)}$, we are led to the relation (1.5), which we restate as the following theorem.

Theorem 3.2 *For $n \geq 1$, we have $|A_{2n}(4312)| = |A_{2n}(1234)|$.*

Let us first recall some notation and terminology. A sequence $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$ of positive integers is said to be a partition of n if $n = \lambda_1 + \lambda_2 + \dots + \lambda_m$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m > 0$, where each λ_i is called a part of λ . A Young diagram of shape λ is defined to be a left-justified array of n boxes with λ_1 boxes in the first row, λ_2 boxes in the second row and so on. If λ is a partition with distinct parts, then the shifted Young diagram of shape λ is an array of cells with m rows, where each row is indented by one cell to the right with respect to the previous row, and there are λ_i cells in row i .

A standard Young tableau of shape λ is a Young diagram of λ whose boxes have been filled with the number $1, 2, \dots, n$ such that the entries are increasing along each row and each column. A shifted standard Young tableau of shape λ is a filling of a shifted Young diagram with $1, 2, \dots, n$ such that the entries are increasing along each row and each column. We denote by $SHSYT(\lambda)$ the set of shifted standard Young tableaux of shape λ .

As shown by Lewis [6], $A_{2n}(1234)$ is enumerated by the n -th 3-dimensional Catalan number $C_n^{(3)}$ as given in (1.1). To prove relation (1.5), it suffices to demonstrate that $A_{2n}(4312)$ is also counted by $C_n^{(3)}$. In light of the correspondence between 4312-avoiding alternating permutations and 1243-avoiding down-up permutations via complement, we proceed to consider the generating tree for $A_{2n+1}(1243)$.

It turns out that the generating tree for $A_{2n+1}(1243)$ is isomorphic to the generating tree for $SHSYT(n+2, n+1, n)$ obtained by Lewis [6] as given below:

$$\begin{cases} \text{root:} & (1, 2), \\ \text{rule:} & (a, b) \mapsto \{(x, y) \mid 2 \leq x \leq a+1 \text{ and } x+1 \leq y \leq b+2\}. \end{cases} \quad (3.1)$$

The above generating tree is based on the following labeling scheme. Let T be a shifted standard Young tableau $T \in SHSYT(n+2, n+1, n)$, and let $T(i, j)$ denote the entry of T in the i -th row and the j -th column. We associate T with a label (a, b) , where $a = 3n+4 - T(2, n+2)$ and $b = 3n+4 - T(1, n+2)$. The isomorphism can be easily established by mapping a label (a, b) in (3.1) to a label $(a-1, b)$ in (2.5). This leads to the relation

$$|A_{2n+1}(1243)| = |SHSYT(n+2, n+1, n)|. \quad (3.2)$$

Observe that in the constructions of the generating tree for $A_{2n+1}(1243)$ and the generating tree for $SHSYT(n+2, n+1, n)$, the children of any alternating permutation

and any shifted standard Young tableau are uniquely labeled. So the above isomorphism between the generating trees of $A_{2n+1}(1243)$ and $SHSYT(n+2, n+1, n)$ can be restricted to certain classes of labels. More precisely, we have the following correspondence.

Theorem 3.3 *For $n \geq 1$, there is a one-to-one correspondence between the set P_n of alternating permutations in $A_{2n+1}(1243)$ with labels of the form $(1, b)$ and the set Q_n of shifted standard Young tableaux in $SHSYT(n+2, n+1, n)$ with labels of the form $(2, b)$.*

By the labeling schemes, we have

$$P_n = \{u \mid u \in A_{2n+1}(1243), 2n+2 - \max\{u_1 + 1, f(u) + 1\} = 1\}, \quad (3.3)$$

$$Q_n = \{T \mid T \in SHSYT(n+2, n+1, n), T(2, n+2) = 3n+2\}. \quad (3.4)$$

Based on the relation $|P_n| = |Q_n|$, we aim to prove Theorem 3.1. To this end, we shall give characterizations of P_n and Q_n without using the labels. Then we shall give a bijection between the set P_n and the set $A_{2n}(4312) \cup A_{2n-1}(1243)$ and a bijection between the set Q_n and the set

$$SHSYT(n+1, n, n-1) \cup SHSYT(n+2, n, n-2).$$

In view of (3.2), we see that $|A_{2n-1}(1243)| = |SHSYT(n+1, n, n-1)|$. Thus we arrive at Theorem 3.1.

Lemma 3.4 *For $n \geq 0$, an alternating permutation $u = u_1 u_2 \cdots u_{2n+1} \in A_{2n+1}(1243)$ is in P_n if and only if $u_2 = 2n+1$, that is,*

$$P_n = \{u \mid u = u_1 u_2 \cdots u_{2n+1} \in A_{2n+1}(1243), u_2 = 2n+1\}. \quad (3.5)$$

Proof. Assume that $u = u_1 u_2 \cdots u_{2n+1} \in P_n$, that is, $2n+2 - \max\{u_1 + 1, f(u) + 1\} = 1$. It follows that $u_1 = 2n$ or $f(u) = 2n$. If $u_1 = 2n$, then we have $u_1 < u_2 \leq 2n+1$ since $u \in A_{2n+1}(1243)$. This implies that $u_2 = 2n+1$. If $f(u) = 2n$, by the definition of $f(u)$, we find that $2n+1$ precedes $2n$ in u , since $(2n+1)(2n)$ is the only 21-pattern in u with $2n$ being a smaller element. We aim to show that $u_2 = 2n+1$. Assume to the contrary that $u_2 < 2n+1$. Then $u_1 u_2 (2n+1)(2n)$ forms a 1243-pattern of u since $u_1 < u_2$. So we have $u_2 = 2n+1$.

Conversely, assume that $u \in A_{2n+1}(1243)$ and $u_2 = 2n+1$. We wish to show that $2n+2 - \max\{u_1 + 1, f(u) + 1\} = 1$. We consider two cases. If $u_1 = 2n$, then by the definition of $f(u)$, we have $f(u) < 2n$. It follows that $2n+2 - \max\{u_1 + 1, f(u) + 1\} = 1$. If $u_1 \neq 2n$, then $(2n+1)(2n)$ forms a 21-pattern of u since $u_2 = 2n+1$. From the definition of $f(u)$ it can be seen that $f(u) = 2n$. Hence we also get $2n+2 - \max\{u_1 + 1, f(u) + 1\} = 1$. This completes the proof. \blacksquare

Theorem 3.5 *For $n \geq 1$, there is a bijection between P_n and $A_{2n}(4312) \cup A_{2n-1}(1243)$.*

Proof. We divide P_n into two subsets P'_n and P''_n , where

$$P'_n = \{u \mid u = u_1 u_2 \cdots u_{2n+1} \in A_{2n+1}(1243), u_2 = 2n+1 \text{ and } u_1 > u_3\},$$

$$P''_n = \{u \mid u = u_1 u_2 \cdots u_{2n+1} \in A_{2n+1}(1243), u_2 = 2n+1 \text{ and } u_1 < u_3\}.$$

We proceed to show that there is a bijection between P'_n and $A_{2n}(4312)$ and there is a bijection between P''_n and $A_{2n-1}(1243)$.

First, we define a map $\varphi: P'_n \rightarrow A_{2n}(4312)$. Given $v = v_1 v_2 \cdots v_{2n+1} \in P'_n$, let $\varphi(v) = \pi^c$, where $\pi = v_1 v_3 v_4 \cdots v_{2n+1}$. Note that π is a permutation of $[2n]$ since $v_2 = 2n+1$. Moreover, it can be seen that π is 1243-avoiding since v is 1243-avoiding and π is a subsequence of v . From the fact $v_1 > v_3 < v_4 > \cdots > v_{2n+1}$, we see that π is a down-up permutation. It follows that $\pi \in A'_{2n}(1243)$. Hence we deduce that $\varphi(v) = \pi^c \in A_{2n}(4312)$. That is to say, φ is well-defined.

To prove that φ is a bijection, we construct the inverse of φ . Define a map $\phi: A_{2n}(4312) \rightarrow P'_n$. Given $w = w_1 w_2 \cdots w_{2n} \in A_{2n}(4312)$, let $\phi(w) = \tau = (2n+1 - w_1)(2n+1)(2n+1 - w_2) \cdots (2n+1 - w_{2n})$. We claim that τ is 1243-avoiding. Since w is 4312-avoiding, by complement we see that $\tau_1 \tau_3 \tau_4 \cdots \tau_{2n+1}$ is 1243-avoiding. Note that $\tau_2 = 2n+1$ does not occur in any 1243-pattern of τ . So the claim is verified. Evidently, τ is alternating, and hence we have $\tau \in A_{2n+1}(1243)$. From the fact that $w_1 < w_2$ we see that $\tau_1 > \tau_3$. Thus $\tau \in P'_n$, and so ϕ is well-defined. Moreover, it can be easily checked that $\phi = \varphi^{-1}$. So we conclude that φ is a bijection between P'_n and $A_{2n}(4312)$.

We next construct a bijection between P''_n and $A_{2n-1}(1243)$. Given an alternating permutation $u = u_1 u_2 \cdots u_{2n+1}$ in P''_n , define $\psi(u) = st(r)$, where $r = u_1 u_3 u_5 u_6 \cdots u_{2n} u_{2n+1}$ and $st(r)$ is the permutation of $[2n-1]$ which is order isomorphic to r .

We claim that ψ is well-defined, that is, $\psi(u)$ is an alternating permutation in $A_{2n-1}(1243)$. Since $u \in P''_n$, we find that $u_1 < u_3 < u_4$ and $u_2 = 2n+1$. We assert that $u_3 + 1 = u_4$. Otherwise, $u_1 u_3 u_4 (u_3 + 1)$ would form a 1243-pattern of u , contradicting to the fact u is 1243-avoiding. Since $u_4 > u_5$, we deduce that $u_3 > u_5$. It follows that $u_1 < u_3 > u_5 < \cdots < u_{2n} > u_{2n+1}$. Thus $\psi(u)$ is an alternating permutation of length $2n-1$. It is clear that $\psi(u)$ is 1243-avoiding since u is 1243-avoiding and u contains $\psi(u)$ as a pattern. So we conclude that $\psi(u) \in A_{2n-1}(1243)$. This proves the claim.

To prove that ψ is a bijection, we describe the inverse of ψ . Given $q = q_1 q_2 \cdots q_{2n-1}$ in $A_{2n-1}(1243)$, define $\theta(q) = p$, where $p = p_1 p_2 \cdots p_{2n+1}$ is obtained from q by inserting $2n+1$ after q_1 and inserting $q_2 + 1$ after q_2 , and increasing each element of q which is not less than $q_2 + 1$ by 1. For example, for $q = 34152 \in A_5(1243)$, we have $p = \theta(q) = 3745162$.

We need to show that θ is well-defined, that is, p is an alternating permutation in P''_n . By the construction of p , we have $p_1 = q_1$, $p_2 = 2n+1$, $p_3 = q_2$, $p_4 = q_2 + 1$ and $p_5 = q_3$. It follows that $p_1 < p_2 > p_3 < p_4 > p_5$. Since $q_3 q_4 \cdots q_{2n-1}$ is order isomorphic to $p_5 p_6 \cdots p_{2n+1}$, we find that $p_5 < p_6 > \cdots > p_{2n+1}$. This proves that p is alternating.

We proceed to show that p is 1243-avoiding. Assume to the contrary that $p_i p_j p_k$ forms a 1243-pattern of p . Since q is 1243-avoiding, from the construction of p , we see that $p_i p_j p_k$ must contain p_2 or p_4 . Since $p_2 = 2n + 1$ cannot occur in any 1243-pattern, p_4 appears in $p_i p_j p_k$. Moreover, p_3 must appear in $p_i p_j p_k$. Otherwise, we assume that p_3 does not appear in $p_i p_j p_k$. Since $p_3 = q_2$ and $p_4 = q_2 + 1$, we see that $p_3 + 1 = p_4$. By replacing p_4 with p_3 in $p_i p_j p_k$ we obtain a 1243-pattern which does not contain p_4 , a contradiction. So we have shown that $p_i p_j p_k$ contains both p_3 and p_4 .

Since $p_3 + 1 = p_4$, we have $p_3 < p_4$. By the assumption that $p_i p_j p_k$ forms a 1243-pattern, we have either $p_i p_j = p_3 p_4$ or $p_i p_k = p_3 p_4$. If $p_i p_j = p_3 p_4$, that is, $p_3 p_4 p_j p_k$ is a 1243-pattern, where $j > 4$. Then $p_1 p_3 p_j p_k$ forms a 1243-pattern since $p_1 < p_3 < p_4$, contradicting to the assertion that p_4 must appear in any 1243-pattern of p . We now consider the case $p_i p_k = p_3 p_4$, namely, $p_i p_3 p_4 p_k$ is a 1243-pattern, where $k > 4$. This yields that $p_3 < p_k < p_4$. But this is impossible because $p_3 + 1 = p_4$. This proves that p is 1243-avoiding.

Till now, we have shown that $p \in A_{2n+1}(1243)$. Combining the fact that $p_2 = 2n + 1$ and $p_1 < p_3$, we see that $p \in P_n''$. It follows that θ is a well-defined map from $A_{2n-1}(1243)$ to P_n'' . It is easy to verify that $\theta = \psi^{-1}$. Hence ψ is a bijection between P_n'' and $A_{2n-1}(1243)$. This completes the proof. \blacksquare

Theorem 3.6 *For $n \geq 3$, there is a bijection between Q_n and*

$$SHSYT(n+1, n, n-1) \cup SHSYT(n+2, n, n-2).$$

Proof. We first decompose Q_n into two subsets Q'_n and Q''_n , where

$$\begin{aligned} Q'_n &= \{T \mid T \in Q_n, T(3, n+1) = 3n+1\}, \\ Q''_n &= \{T \mid T \in Q_n, T(1, n+2) = 3n+1\}. \end{aligned}$$

Clearly, $Q'_n \cap Q''_n = \emptyset$. We wish to show that $Q_n = Q'_n \cup Q''_n$. It suffices to prove that $Q_n \subseteq Q'_n \cup Q''_n$.

Given a shifted standard Young tableau T in Q_n , since $T(2, n+2) = 3n+2$ and $T(2, n+2) < T(3, n+2) \leq 3n+3$, we find that $T(3, n+2) = 3n+3$. Since the entries in a shifted standard Young tableau are increasing along each row and each column, we have $T(1, n+2) = 3n+1$ or $T(3, n+1) = 3n+1$. So we deduce that $Q_n \subseteq Q'_n \cup Q''_n$.

We now define a map χ from Q_n to the set $SHSYT(n+1, n, n-1) \cup SHSYT(n+2, n, n-2)$. Let T be a shifted standard Young tableau in Q_n . If $T \in Q'_n$, then let $\chi(T) = T_1$, where T_1 is obtained from T by deleting the boxes $T(2, n+2)$, $T(3, n+1)$ and $T(3, n+2)$. If $T \in Q''_n$, then let $\chi(T) = T_2$, where T_2 is obtained from T by deleting the boxes $T(1, n+2)$, $T(2, n+2)$ and $T(3, n+2)$. It is easy to verify that χ is well-defined and it is a bijection between Q_n and $SHSYT(n+1, n, n-1) \cup SHSYT(n+2, n, n-2)$. This completes the proof. \blacksquare

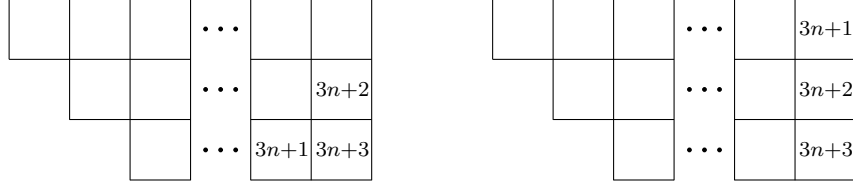


Figure 3.1: The two cases when $T(2, n+2) = 3n+2$

Figure 3.1 gives an illustration of the two cases when $T(2, n+2) = 3n+2$.

It is clear that Theorem 3.1 follows from Theorem 3.5 and Theorem 3.6. Recall that $|SHSYT(n+2, n, n-2)|$ is counted by the n -th 3-dimensional Catalan number $C_n^{(3)}$. On the other hand, Lewis [6] has shown that $|A_{2n}(1234)| = C_n^{(3)}$. Thus for $n \geq 3$, we have $|A_{2n}(4312)| = |A_{2n}(1234)|$. Note that this relation also holds for $n = 1, 2$. This completes the proof of Theorem 3.2.

4 A generating tree for $A_{2n}(4312)$

In this section, we construct a generating tree for $A_{2n}(4312)$. While this generating tree is not isomorphic to that for $A_{2n}(1234)$ given by Lewis [6], it allows us to give a second proof of Theorem 2.1, namely, $|A_{2n+1}(1243)| = |A_{2n+1}(2143)|$. To be more specific, by deleting the leaves of the generating tree for $A_{2n}(4312)$ and changing every label (a, b) to $(a-1, b)$, we are led to the generating tree for $A_{2n}(3412)$ as given by Lewis [6]. By restricting this correspondence to certain labels, we obtain Theorem 2.1.

We now present the construction of the generating tree for $A_{2n}(4312)$, which is analogous to the construction of the generating tree for $A_{2n+1}(1243)$ as given in Section 2. First, we characterize the set of 4312-avoiding alternating permutations in A_{2n+2} that are generated by an alternating permutation u in $A_{2n}(4312)$.

Theorem 4.1 *For $n \geq 1$, given a permutation $u = u_1 u_2 \cdots u_{2n} \in A_{2n}(4312)$, define*

$$g(u) = \min\{2n+1, u_i \mid \text{there exist } j \text{ and } k \text{ such that } i < j < k \text{ and } u_j < u_k < u_i\}.$$

Then w is a child of u if and only if it is of the form $w = v_1 \mapsto (v_2 \mapsto u)$, where

$$1 \leq v_1 \leq v_2. \tag{4.1}$$

and

$$u_1 + 1 \leq v_2 \leq g(u), \tag{4.2}$$

Proof. Suppose that $w = w_1w_2 \cdots w_{2n+2}$ is a child of u , that is, w is of the form $v_1 \mapsto (v_2 \mapsto u)$ and $w \in A_{2n+2}(4312)$. Since w is alternating on $[2n+2]$, we see that $1 \leq v_1 \leq v_2 \leq 2n+1$ and $v_2 \geq u_1 + 1$. Moreover, we claim that $v_2 \leq g(u)$. Otherwise, there exist $i < j < k$ such that $u_j < u_k < u_i$ and $v_2 > u_i$. By the construction of w , we find that $w_2w_{i+2}w_{j+2}w_{k+2}$ forms a 4312-pattern of w , a contradiction. Hence we are led to the relations (4.1) and (4.2).

Conversely, suppose that v_1 and v_2 are integers satisfying conditions (4.1) and (4.2). To prove that $w = v_1 \mapsto (v_2 \mapsto u) = w_1w_2 \cdots w_{2n+2}$ is a child of u , it suffices to show that w is an alternating permutation in $A_{2n+2}(4312)$. Clearly, w is alternating, since $v_2 \geq u_1 + 1$ and $1 \leq v_1 \leq v_2$,

It remains to show that w is 4312-avoiding. Otherwise, we may assume that $w_tw_iw_jw_k$ is a 4312-pattern. We claim that we can always choose $t = 2$. Since u is 4312-avoiding, by the construction of w , we see that $w_tw_iw_jw_k$ contains either w_1 or w_2 . If $w_1w_iw_jw_k$ is a 4312-pattern, since $w_1 < w_2$ we find that $w_2w_iw_jw_k$ is also a 4312-pattern. So the claim is valid. We continue to prove that $w_2w_iw_jw_k$ cannot be a 4312-pattern.

Let $s = v_2 \mapsto u$ and write $s = s_1s_2 \cdots s_{2n+1}$. Clearly, $s_1 = v_2$ and $s_1s_{i-1}s_{j-1}s_{k-1}$ is a 4312-pattern of s , since $w_2w_iw_jw_k$ is assumed to be a 4312-pattern. It follows that $v_2 > s_{i-1} = u_{i-2}$. Note that $u_{i-2}u_{j-2}u_{k-2}$ is a 312-pattern of u . By the definition of $g(u)$, it can be checked that $u_{i-2} \geq g(u)$. So we get $v_2 > u_{i-2} \geq g(u)$, contradicting to the condition (4.2). Hence $w_2w_iw_jw_k$ cannot be a 4312-pattern. This implies that w is 4312-avoiding. So we conclude that w is an alternating permutation in $A_{2n+2}(4312)$, that is to say that w is a child of u . This completes the proof. ■

Notice that using the above generating scheme, some permutations in $A_{2n}(4312)$ do not have any children. Such permutations are called leaves of the generating tree. Permutations having at least one child are called internal vertices of the generating tree. For example, the alternating permutation $3412 \in A_4(4312)$ is a leaf and the alternating permutation $23154867 \in A_8(4312)$ is an internal vertex.

The following two theorems give characterizations of leaves and internal vertices of the generating tree for $A_{2n}(4312)$.

Theorem 4.2 *For $n \geq 1$, given a permutation $u = u_1u_2 \cdots u_{2n} \in A_{2n}(4312)$, u is a leaf if and only if $g(u) = u_1$.*

Proof. We assume that u is a leaf, namely, u has no child. By Theorem 4.1, we see that if u has a child, then it is of the form $v_1 \mapsto (v_2 \mapsto u)$ satisfying conditions (4.1) and (4.2), namely, $1 \leq v_1 \leq v_2$ and $u_1 + 1 \leq v_2 \leq g(u)$. Now that u has no child, there does not exist integers v_1 and v_2 satisfying (4.1) and (4.2). It follows that $u_1 + 1 > g(u)$. Moreover, we claim that $u_1 \leq g(u)$. Otherwise, there exist $i < j < k$ such that $u_j < u_k < u_i$ and $u_1 > u_i$. Consequently, $u_1u_iu_ju_k$ forms a 4312-pattern of u , which contradicts to the fact that u is 4312-avoiding. So the claim is justified. Thus, we have $u_1 \leq g(u) < u_1 + 1$, namely, $g(u) = u_1$.

Conversely, assume that $g(u) = u_1$. By Theorem 4.1, it can be easily verified that u has no child. This completes the proof. ■

Theorem 4.3 *For $n \geq 1$, given a permutation $u = u_1u_2 \cdots u_{2n} \in A_{2n}(4312)$, define*

$$h(u) = \min\{2n + 1, u_j \mid \text{there exists } i \text{ such that } i < j \text{ and } u_i < u_j\}.$$

Then u is an internal vertex if and only if $h(u) = u_1 + 1$.

Proof. Assume that u is an internal vertex. We claim that $u_1 \leq h(u)$. Otherwise, we assume that $u_1 > h(u)$. Then there exist $i < j$ such that $u_i < u_j$ and $u_1 > u_j$. It follows that $u_1u_iu_j$ forms a 312-pattern of u . By the definition of $g(u)$, we have $g(u) \leq u_1$. Meanwhile, since u is 4312-avoiding, we find that $u_1 \leq g(u)$. Thus, we reach the equality $u_1 = g(u)$. By Theorem 4.2, this implies that u is a leaf, a contradiction. Hence the claim is verified.

Observe that u_1 cannot be the second entry of any 12-pattern. So by the definition of $h(u)$, it can be checked that $u_1 \neq h(u)$. It follows that $u_1 < h(u)$. On the other hand, since $u_1(u_1 + 1)$ is a 12-pattern, by the definition of $h(u)$, it can be seen that $h(u) \leq u_1 + 1$. In summary, we obtain $u_1 < h(u) \leq u_1 + 1$, namely, $h(u) = u_1 + 1$.

Conversely, assume that $h(u) = u_1 + 1$. We claim that $h(u) \leq g(u)$. If $g(u) = 2n + 1$, then it is clear that $h(u) \leq g(u)$. If $g(u) < 2n + 1$, then there exist $i < j < k$ such that $u_j < u_k < u_i$ and $g(u) = u_i$. By the definition of $h(u)$, we see that $h(u) \leq u_k$. Thus we have $h(u) \leq u_k < u_i = g(u)$. It follows that for both cases we have $h(u) \leq g(u)$, and so the claim is justified. By the assumption that $h(u) = u_1 + 1$, we obtain $u_1 + 1 \leq g(u)$. By Theorem 4.1, the set of children of u is nonempty. So u is an internal vertex. This completes the proof. ■

To construct the generating tree, we now give a labeling scheme for alternating permutations in $A_{2n}(4312)$. For $n \geq 1$, given a permutation $u = u_1u_2 \cdots u_{2n} \in A_{2n}(4312)$, if u is a leaf, we associate it with a label $(0, 0)$. If u is an internal vertex, we associate it with a label $(h(u), g(u))$. For example, let $u = 46253817 \in A_8(4312)$. Since $g(u) = u_1 = 4$, by Theorem 4.2, we see that u is a leaf. Hence the label of u is $(0, 0)$. It is easily seen that $12 \in A_2(4312)$ is an internal vertex and it has a label $(2, 3)$.

The above labeling scheme enables us to give a characterization of the labels of the children generated by u . Like the extensions of the functions $f(u)$ and $e(u)$ defined in Section 2 to finite integer sequences, the functions $g(u)$ and $h(u)$ can also be extended to finite integer sequences.

Theorem 4.4 *Assume that $u = u_1u_2 \cdots u_{2n}$ is an alternating permutation in $A_{2n}(4312)$ with label (a, b) . If u is an internal vertex, then it generates $\binom{b-a+1}{2}$ leaves and the set of labels of the internal vertices generated by u is given by the set*

$$\{(x, y) \mid 2 \leq x \leq a + 1, a + 2 \leq y \leq b + 2\}.$$

Proof. Assume that $w = v_1 \mapsto (v_2 \mapsto u)$ is a child of u and let $w = w_1 w_2 \cdots w_{2n+2}$. We aim to characterize the label (x, y) of w . According to Theorem 4.1, we have relations (4.1) and (4.2), namely, $1 \leq v_1 \leq v_2$ and $u_1 + 1 \leq v_2 \leq g(u)$. Since u is an internal vertex, it follows from Theorem 4.3 that $u_1 + 1 = h(u)$. Hence relation (4.2) is equivalent to $h(u) \leq v_2 \leq g(u)$.

By the labeling scheme, we see that if w is a leaf, then $(x, y) = (0, 0)$. If w is an internal vertex, then $(x, y) = (h(w), g(w))$. In order to determine the range of (x, y) , we distinguish the case when w is a leaf and the case when w is an internal vertex. We shall derive expressions of $h(w)$ and $g(w)$ in terms of v_1, v_2 and the label (a, b) .

Let $s = w_3 w_4 \cdots w_{2n+2}$. By the same argument as in the proof of Theorem 2.4, we see that in order to determine $h(w)$, it suffices to compare $h(s)$ with the larger element of each new 12-pattern caused by the insertions of v_1 and v_2 into u . The computation of $g(w)$ can be carried out in the same manner. Here are three cases.

Case 1: $h(u) + 1 \leq v_1 \leq v_2, h(u) + 1 \leq v_2 \leq g(u)$. By the construction of w , we see that $w_1 = v_1$ and $w_2 = v_2 + 1$. We proceed to judge w is a leaf or an internal vertex. To this end, we compute $g(w)$. Since both v_1 and v_2 are not larger than $g(u)$ and s is order isomorphic to u , it can be easily verified that $g(s) = g(u) + 2$.

Now we consider the newly formed 312-patterns caused by the insertions of v_1 and v_2 into u . By the assumption that $h(u) + 1 \leq v_1 \leq v_2$, there exist $i < j$ such that $v_1 > u_j > u_i$. It is clear that $w_1 w_{i+2} w_{j+2} = v_1 u_i u_j$. Thus $w_1 w_{i+2} w_{j+2}$ forms a 312-pattern of w . It is easily verified that v_1 is the smallest entry among the largest elements of the newly formed 312-patterns. Since $v_1 \leq g(u)$, we deduce that $g(w) = \min\{v_1, g(s)\} = v_1 = w_1$. By Theorem 4.2, we see that w is a leaf. Hence in this case u only generates leaves. Using the labeling scheme for $A_{2n}(4312)$, we obtain that $a = h(u)$ and $b = g(u)$. So the number of leaves generated by u is given by

$$\sum_{v_2=a+1}^b (v_2 - a) = 1 + 2 + \cdots + (b - a) = \binom{b - a + 1}{2}.$$

Case 2: $1 \leq v_1 \leq h(u), h(u) + 1 \leq v_2 \leq g(u)$. Since s is order isomorphic to u , using the same argument as in the proof of Theorem 2.4, we obtain that $h(s) = h(u) + 1$ and $g(s) = g(u) + 2$. To compute $h(w)$, we consider the newly formed 12-patterns caused by the insertions of v_1 and v_2 into u . First, $v_1(v_1 + 1)$ is a newly formed 12-pattern in w . Moreover, it can be seen that $v_1 + 1$ is the minimal entry among the larger elements in the newly formed 12-patterns. Notice that $v_1 + 1 \leq h(u) + 1 = h(s)$. So we have $h(w) = \min(h(s), v_1 + 1) = v_1 + 1$. By Theorem 4.3, we find that w is an internal vertex.

To determine the range of the label of w , it suffices to compute $g(w)$. Let us consider the newly formed 312-patterns in w . Since $w_1 = v_1 \leq h(u)$, we see that w_1 cannot occur in any 312-pattern of w . Since $v_2 \geq h(u) + 1$, we deduce that $w_2 = v_2 + 1$ is the largest entry of a 312-pattern in w . From the fact that $v_2 + 1 < g(u) + 2 = g(s)$ we obtain that $g(w) = \min(v_2 + 1, g(s)) = v_2 + 1$. Therefore, the label of w is given by

$(x, y) = (v_1 + 1, v_2 + 1)$. By the labeling scheme, we see that $a = h(u)$ and $b = g(u)$. From the assumption of this case, we get $2 \leq x \leq a + 1$ and $a + 2 \leq y \leq b + 1$. This implies that the set of labels of the children of u considered in this case is given by

$$\{(x, y) \mid 2 \leq x \leq a + 1, a + 2 \leq y \leq b + 1\}.$$

Case 3: $1 \leq v_1 \leq h(u), v_2 = h(u)$. Since s is order isomorphic to u , using the same argument as in the proof of Theorem 2.4, we obtain that $h(s) = h(u) + 2$. Notice that $w_1(w_1 + 1)$ is a 12-pattern of w and $w_1 + 1$ is the minimal entry of the larger elements in the newly formed 12-patterns caused by the insertions of v_1 and v_2 . Since $w_1 = v_1 \leq h(u)$, we find that $h(w) = \min(w_1 + 1, h(s)) = \min(w_1 + 1, h(u) + 2) = w_1 + 1$. According to Theorem 4.3, w is an internal vertex.

It remains to determine $g(w)$. Recall that $h(u) \leq g(u)$. Hence in this case we have $v_1 \leq v_2 \leq g(u)$. By the reasoning in the proof of Theorem 2.4, we deduce that $g(s) = g(u) + 2$. Since $v_1 \leq v_2 = h(u)$, we see that neither w_1 nor w_2 can be the largest entry of a 312-pattern of w . This yields that $g(w) = g(s) = g(u) + 2$. Therefore, the label of w is given by $(x, y) = (v_1 + 1, g(u) + 2)$. Since the label (a, b) is given by $a = h(u)$ and $b = g(u)$, by the assumption that $1 \leq v_1 \leq h(u)$ and $v_2 = h(u)$, we obtain that $2 \leq x \leq a + 1$ and $y = b + 2$. It follows that the set of labels of the children of u considered in this case is given by

$$\{(x, y) \mid 2 \leq x \leq a + 1, y = b + 2\}.$$

Combining the above three cases, we see that an internal vertex u generates $\binom{b-a+1}{2}$ leaves and $a(b - a + 1)$ internal vertices labeled by (x, y) , where $2 \leq x \leq a + 1$ and $a + 2 \leq y \leq b + 2$. This completes the proof. \blacksquare

By Theorem 4.4, the generating tree for $A_{2n}(4312)$ can be described as follows:

$$\begin{cases} \text{root:} & (2, 3), \\ \text{rule:} & (a, b) \mapsto \{(x, y) \mid 2 \leq x \leq a + 1 \text{ and } a + 2 \leq y \leq b + 2\} \\ & \cup \binom{b-a+1}{2} \text{ occurrences of } (0, 0). \end{cases}$$

For $n \geq 1$, if we restrict our attention to the internal vertices in $A_{2n}(4312)$, then we are led to the following generating tree:

$$\begin{cases} \text{root:} & (2, 3), \\ \text{rule:} & (a, b) \mapsto \{(x, y) \mid 2 \leq x \leq a + 1 \text{ and } a + 2 \leq y \leq b + 2\}. \end{cases} \quad (4.3)$$

Indeed, the above generating tree is isomorphic to the generating tree for $A_{2n}(3412)$ as given by Lewis [6]:

$$\begin{cases} \text{root:} & (1, 3), \\ \text{rule:} & (a, b) \mapsto \{(x, y) \mid 1 \leq x \leq a + 1 \text{ and } a + 3 \leq y \leq b + 2\}. \end{cases} \quad (4.4)$$

The one-to-one correspondence is easily established by mapping a label (a, b) in (4.3) to a label $(a - 1, b)$ in (4.4). By restricting this correspondence to certain labels, we arrive at the following bijection.

Theorem 4.5 *There is a one-to-one correspondence between the set U_n of alternating permutations in $A_{2n}(4312)$ with labels of the form $(2, b)$ and the set V_n of alternating permutations in $A_{2n}(3412)$ with labels of the form $(1, b)$.*

The above theorem leads to an alternative proof of Theorem 2.1, that is, for $n \geq 0$, we have $|A_{2n+1}(1243)| = |A_{2n+1}(2143)|$. To this end, we give characterizations of U_n and V_n without using labels.

Theorem 4.6 *For $n \geq 1$, we have*

$$U_n = \{u \mid u = u_1 u_2 \cdots u_{2n} \in A_{2n}(4312), u_1 = 1\}, \quad (4.5)$$

$$V_n = \{u \mid u = u_1 u_2 \cdots u_{2n} \in A_{2n}(3412), u_{2n} = 2n\}. \quad (4.6)$$

Proof. Recall that for a permutation $w \in A_{2n}(3412)$ with label (a, b) in the generating tree defined by Lewis [6], we have $a = d(w)$, where

$$d(w) = 2n - \max\{w_i \mid \text{there exists } j \text{ such that } j > i \text{ and } w_i < w_j\}. \quad (4.7)$$

By Theorem 4.3, a permutation $u = u_1 u_2 \cdots u_{2n} \in A_{2n}(4312)$ is an internal vertex if and only if $h(u) = u_1 + 1$. Using the labeling schemes for $A_{2n}(4312)$ and $A_{2n}(3412)$, we find that U_n and V_n can be described in terms of the functions $h(u)$ and $d(u)$, namely,

$$U_n = \{u \mid u = u_1 u_2 \cdots u_{2n} \in A_{2n}(4312), h(u) = u_1 + 1 \text{ and } h(u) = 2\}, \quad (4.8)$$

$$V_n = \{u \mid u = u_1 u_2 \cdots u_{2n} \in A_{2n}(3412), d(u) = 1\}. \quad (4.9)$$

We first prove (4.5). Given $u = u_1 u_2 \cdots u_{2n} \in U_n$, it is easily seen that $u_1 = 1$. Conversely, assume that $u = u_1 u_2 \cdots u_{2n}$ is an alternating permutation in $A_{2n}(4312)$ with $u_1 = 1$. Since the subsequence 12 forms a 12-pattern of u , by the definition of $h(u)$, we obtain that $h(u) = 2$. Thus the relation $h(u) = u_1 + 1$ holds. It follows that $u \in U_n$. Hence (4.5) is verified.

We now consider (4.6). Assume that $u = u_1 u_2 \cdots u_{2n}$ is an alternating permutation in V_n . Since $d(u) = 1$, we see that $\max\{u_i \mid \text{there exists } j \text{ such that } j > i \text{ and } u_i < u_j\} = 2n - 1$. Notice that $(2n - 1)(2n)$ is the only 12-pattern of u with $2n - 1$ being a smaller element. It follows that $2n - 1$ precedes $2n$ in u . If $u_{2n} \neq 2n$, then $(2n - 1)(2n)u_{2n-1}u_{2n}$ forms a 3412-pattern of u , which is a contradiction. Thus we have $u_{2n} = 2n$. Conversely, if $u_{2n} = 2n$, it is easily seen that $d(u) = 1$. This completes the proof. ■

In view of Theorem 4.5, we see that $|U_n| = |V_n|$. To prove Theorem 2.1, we shall give a bijection between U_n and $A_{2n-1}(1243)$ and a bijection between V_n and $A_{2n-1}(3412)$. Hence Theorem 2.1 follows from the fact $|A_{2n-1}(2143)| = |A_{2n-1}(3412)|$.

Define a map $\rho: U_n \rightarrow A_{2n-1}(1243)$ as follows. Given an alternating permutation $w = w_1 w_2 \cdots w_{2n} \in U_n$, let $\rho(w) = \pi^c$, where $\pi = (w_2-1)(w_3-1) \cdots (w_{2n}-1)$. Obviously, π is a 4312-avoiding down-up permutation on $[2n-1]$. It follows that $\rho(w) \in A_{2n-1}(1243)$. Thus, ρ is well-defined. Using the same argument as in the proof of Lemma 3.5, it can be shown that ρ is a bijection.

To define a map $\mu: V_n \rightarrow A_{2n-1}(3412)$, as we assume that u is an alternating permutation in V_n . Let $\mu(u)$ be the alternating permutation obtained from u by deleting the last element. It is easy to verify that μ is a bijection. This gives an alternative proof of Theorem 2.1.

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